SU-ITP-00/01
hep-th/0001072
January 17, 2000

Twistors and Actions on Coset Manifolds

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Abstract

Particle and string actions on coset spaces typically lack a quadratic kinetic term, making their quantization difficult. We define a notion of twistors on these spaces, which are hypersurfaces in a vector space that transform linearly under the isometry group of the coset. By associating the points of the coset space with these hypersurfaces, and the internal coordinates of these hypersurfaces with momenta, it is possible to construct manifestly symmetric actions with leading quadratic terms. We give a general algorithm and work out the case of a particle on AdS_p explicitly. In this case, the resulting action is a world-line gauge theory with sources, (the gauge group depending on p) which is equivalent to a nonlocal world-line σ-model.

1 Introduction

The standard action for a particle or string on a coset space G/H is manifestly invariant under G but does not have a quadratic kinetic term. This obstructs the usual quantization procedure. Moreover, the isometries are nonlinearly realized on the coordinates and so even if the action were quadratic, the fields would not automatically form G-representations. This makes it difficult to directly study systems such as strings on $AdS_{p+2} \times S^{d-p-2}$ which are important for understanding holography. [1, 2, 3]

A hint on how to work around this comes from twistors. These were originally set up by Penrose to study conformal Minkowski space, [4] and have since been generalized to conformal superspace [5] and AdS_5. [6, 7] In all of these cases, twistors associated a hypersurface in some vector space which transforms linearly under the isometry group with every point of the coset space. The internal coordinates of these hypersurfaces were associated with momenta and constrained quantities.

This construction can be generalized to arbitrary coset manifolds[1] G/H. A mapping between points of the coset and hypersurfaces in a vector space can be constructed which naturally mimics the geometric structure of the coset. The isometries, for example, can easily be extracted from the linear isometry transformations of the twistors.

If the vector space is also a Hilbert space (i.e. posesses an appropriate inner product) then one can naturally construct objects out of twistors which are manifestly invariant under the coset isometries. Using the twistor mapping, these can be written as (typically fairly complicated) functions of the coset coordinates and the internal coordinates of the twistor. Since these quantities are manifestly G-invariant, one can construct actions out of them which are equivalent to ordinary coset actions if the internal coordinates are identified with momenta. Since the twistor mapping is typically very complicated, very simple twistor actions are equivalent to very complicated coset actions.

We demonstrate this construction for a particle on AdS_p. Twistors are built in a vector space which transforms in the spinor representation of $SO(p-1, 2)$. A worldline gauge theory can be built out of these twistors which is equivalent to the ordinary action for a massive particle on the coset. This theory is equivalent to a nonlocal σ-model whose target space is the vector space. This construction can probably be generalized to the study of particles and strings on anti-de Sitter superspaces such as those important for the AdS/CFT correspondence.

[1]The results below apply both to cosets and supercosets, with no additional restrictions (reductivity, symmetry, etc.) except where explicitly noted.

2 The twistor construction

We begin by describing cosets in a language which naturally leads to twistors. A point in a coset G/H is associated with a hypersurface in the group manifold by the relation

$$\phi(\hat{x} \in G) = \{\hat{x}h : h \in H\} \ . \tag{1}$$

The $\hat{x}$ which generate distinct $\phi(\hat{x})$ are given by

$$\hat{x} := v(x)\hat{x}_0 \ . \tag{2}$$

The x are coordinates on the coset space, $\hat{x}_0$ is a point in G which represents the origin of the coset space, and $v(x)$ is a function from the coordinates into G such that $v(0) = e_G$ and $\phi \circ v$ is 1-1. (i.e., a coset representative[2]) Using this, we can write

$$\phi(x) \equiv \phi(\hat{x}) = v(x)\phi(\hat{x}_0) \ . \tag{3}$$

This maps points of the coordinate space to hypersurfaces in G which are invariant under the right action of H.

The geometry on the coset space is defined by the invariances of the Cartan form $L = v^{-1}dv$. When G is semisimple, this can be contracted with a restricted Cartan-Killing metric to give a metric on the coset. The isometries of this space (which are defined in terms of the vielbein $E = L - L \cdot H$ when there is no metric) are given implicitly by

$$\delta x : \delta v(x) = gv(x) \ , \tag{4}$$

where $g \in G$. This implies that

$$\delta\phi(x) = g\phi(x) \ . \tag{5}$$

Twistors can be thought of as explicit group representations of the ϕ. If we represent the group on a Hilbert space H, the twistor is defined to be a hypersurface in H given by

$$\mathcal{Z}(x) = v(x)\mathcal{Z}_0 \ , \tag{6}$$

where $v(x)$ is the representation of the coset representative and $\mathcal{Z}_0$ is an H-invariant hypersurface in V. (It is the representation of $\phi(x_0)$) The mapping $\mathcal{Z}$ must be 1-1 for the set of $\mathcal{Z}(x)$ to be isomorphic to the coset, which means that the codimension

[2]The canonical form of $v(x)$ is

$$v(x) = e^{x \cdot K}h(x) \ ,$$

where K are the generators of G not in H and $h(x)$ is some element of H chosen to simplify the resulting expression.

of $\mathcal{Z}_0$ must be no less than the dimension of the coset. (codim $\mathcal{Z}(x)$ = codim $\mathcal{Z}_0$ since $v(x)$ is surjective) We express $\mathcal{Z}_0$ as a linear function of some set of internal coordinates.

Although this appears trivial, the twistor construction has two important advantages. First, one can construct manifestly G-invariant quantities out of inner products of various twistors. These quantities are typically complicated functions of the coset coordinates and the internal coordinates of the twistor, whose G-transformations are not obvious. Second, since the $\mathcal{Z}$ are given as explicit functions of the coset coordinates, one can easily extract the geometry of the coset coordinates from the linear isometries of the twistors.

A very simple example of this is the case of conformal Minkowski space ($SO(3,2)/ISO(3,1) \times D$, where D is the dilatation operator) in the 4-component spinor representation of $SO(3,2)$. The initial hypersurface $\mathcal{Z}_0$ has the form

$$\mathcal{Z}_0 = \begin{pmatrix} \lambda_\alpha \\ \mu^{\dot{\alpha}} \end{pmatrix} \tag{7}$$

where the α and $\dot{\alpha}$ are two-component spinor and conjugate spinor indices of $SO(3,1)$, and λ and μ are complex. Lorentz invariance implies that if $\mathcal{Z}_0$ has any point with $\lambda \neq 0$, it must contain all such points, and likewise for μ. Thus the dimension constraint $0 <$ dim $\mathcal{Z}_0 \leq 4$[3] requires that exactly one of the two be free. Without loss of generality, we choose λ. H-invariance ($\delta_H \mathcal{Z}_0 = 0$) then requires $\mu = 0$. Then using the coset representative

$$v(x) = e^{-ix\cdot P} = \begin{pmatrix} 1 & 0 \\ -ix^{\dot{\alpha}\alpha} & 1 \end{pmatrix} \tag{8}$$

the twistor mapping is

$$\mathcal{Z}(x) = \begin{pmatrix} \lambda_\alpha \\ -ix^{\dot{\alpha}\alpha}\lambda_\alpha \end{pmatrix} . \tag{9}$$

This is the familiar Penrose twistor formula. [4] We postpone the discussion of isometries and invariants to the more detailed example of AdS_p below.

It is worth noting the points at which one has freedom of choice in this algorithm. Coset representatives are not unique (they must only guarantee that $\mathcal{Z}(x)$ is a 1-1 function) although there is a canonical choice thereof. The initial hypersurface $\mathcal{Z}_0$ is generally also not unique, but the set of allowable H-invariant hypersurfaces can typically be parametrized. All such choices give equivalent constructions in that

[3] dim $\mathcal{Z}_0 > 0$ since otherwise the twistorization would map points to points and so would be trivial.

they yield the same isometries of the space, although the invariants may depend parametrically on the choice of $\mathcal{Z}_0$.

It will also be important to interpret the internal coordinates of the twistors, (such as λ_α in (9) since invariants will generally depend on these. The physical meaning of these quantities must be determined by physics, specifically by the choice of a twistor action. In a typical case, some combination of these coordinates will be interpreted as a momentum or as a constrained quantity. (This will be demonstrated explicitly below) The twistor mapping assigns a vector space to every point of the coset, and so can be thought of as a bundle. Clearly momenta can only be encoded in this way if the tangent bundle of the coset is a subbundle of the twistor, which requires that the twistor's dimension be at least equal to that of the coset. This condition is satisfied in the case of Penrose twistors; there the action

$$\mathcal{L} = i\bar{\mathcal{Z}}\partial\mathcal{Z} \ , \tag{10}$$

with appropriate twistor metric, is equivalent to the first-order world-line action for a particle in Minkowski space if the momentum is identified with

$$P_{\dot{\alpha}\alpha} = \bar{\lambda}_{\dot{\alpha}}\lambda_\alpha \tag{11}$$

In more general cases the identification is not as straightforward, but typically follows from the choice of action.

3 Twistorization of AdS_p

We now turn to the specific case of particles on $AdS_p = SO(p-1,2)/SO(p-1,1)$. The ordinary world-line action for these particles is manifestly G-invariant but does not have a quadratic kinetic term, so it is useful to try to rephrase this in terms of twistors. This is reasonable since the first-order action,

$$\mathcal{L} = \frac{1}{2}P\cdot\partial x + P_\rho\partial\rho + u\left[\frac{1}{2\rho^2}P^2 - \rho^2 P_\rho^2 - m^2R^2\right] \tag{12}$$

contains only terms of the form $P\cdot\partial x$, which are similar to those found in the conformal Minkowski action (10), and a constraint term which is G-invariant although not manifestly so. In a twistor construction one hopes that this can be rewritten in a manifestly symmetric (and preferably simple) way, and we will see that this is indeed the case.

Twistorization must begin with a choice of G-representation. The two simplest choices are the fundamental and the spinor. The fundamental has simpler group generators, but since its dimension is $(p+1)$ such twistors would have only one internal coordinate and so momenta could not be encoded by the twistor. Therefore we use the spinor representation, which has complex dimension $2^{\lfloor (p+1)/2 \rfloor} \equiv 2d$. The group elements in this representation are[4]

$$g^A{}_B = \begin{pmatrix} L_\beta{}^\alpha + \frac{1}{2} D \delta_\beta{}^\alpha & -iK_{\alpha\dot{\alpha}} \\ -iP^{\dot{\alpha}\alpha} & -\bar{L}^{\dot{\alpha}}{}_{\dot{\beta}} - \frac{1}{2} D \delta^{\dot{\alpha}}{}_{\dot{\beta}} \end{pmatrix} \tag{13}$$

The K and P generate conformal transformations and conformal momentum, which are related to AdS conformal transformations and momenta by

$$\begin{aligned} \tilde{K} &= (K-P)/2 \\ \tilde{P} &= (K+P)/2 \; . \end{aligned} \tag{14}$$

The $L_\beta{}^\alpha$ generate the Lorentz group and the D are dilatations. The stability group is generated by the L's and the $\tilde{K}$'s.

First we must choose an H-invariant initial hypersurface. We can write this surface in the form

$$\mathcal{Z}_0^A = \begin{pmatrix} \lambda_{0\alpha} \\ \mu_0^{\dot{\alpha}} \end{pmatrix} \tag{15}$$

As in the Penrose case, L-invariance requires that if $\mathcal{Z}_0$ contains any independent points with $\lambda_0 \neq 0$, then it contains all such points, and similarly for μ_0. Since we want $0 < \dim \mathcal{Z}_0 \leq 2d-p$, only one of these two should be independent of the other. Without loss of generality we choose λ_0 to be independent, and fix μ_0 by

$$\mu_0^{\dot{\alpha}} = F^{\dot{\alpha}\beta} \lambda_{0\beta} + G^{\dot{\alpha}\dot{\beta}} \bar{\lambda}_{0\dot{\beta}} \tag{16}$$

for some $F^{\dot{\alpha}\beta}$ and $G^{\dot{\alpha}\dot{\beta}}$. $\tilde{K}$-invariance then requires that

$$F\gamma_\mu F + G\gamma_\mu \bar{G} = \gamma_\mu \tag{17}$$

$$F\gamma_\mu G + G\gamma_\mu \bar{F} = 0 \tag{18}$$

where the $\gamma_\mu^{\alpha\dot{\alpha}}$ are the Dirac matrices for $SO(p-2,1)$. For simplicity we will consider twistorizations with $F=0$, so

$$\mathcal{Z}_0^A = \begin{pmatrix} \lambda_{0\alpha} \\ G^{\dot{\alpha}\dot{\beta}} \bar{\lambda}_{0\dot{\beta}} \end{pmatrix} \; . \tag{19}$$

[4]Our index notation is: $\mu = 0 \ldots p-2$ is an $SO(p-2,1)$ (Lorentz) vector index, α, β and $\dot{\alpha}, \dot{\beta} = 1 \ldots d$ are Lorentz spinor and conjugate spinor indices, respectively. A, B and $\dot{A}, \dot{B} = 1 \ldots 2d$ are spinor and conjugate spinor indices of $SO(p-1,2)$.

A natural and simple choice of coset representative is

$$v(x^\mu,\rho)=e^{x\cdot P}e^{D\log\rho}=\begin{pmatrix}\rho^{1/2} & 0\\ -i\rho^{1/2}x^{\dot\alpha\alpha} & \rho^{-1/2}\end{pmatrix}\ ; \tag{20}$$

using this, and defining $\lambda=\rho^{1/2}\lambda_0$, the twistor is

$$\mathcal{Z}^A(x^\mu,\rho)=\begin{pmatrix}\lambda_\alpha\\ -ix^{\dot\alpha\alpha}\lambda_\alpha+\rho^{-1}G^{\dot\alpha\dot\beta}\bar\lambda_{\dot\beta}\end{pmatrix}\ . \tag{21}$$

As a check, the isometries of the space can be calculated from

$$\delta\mathcal{Z}^A=g^A{}_B\mathcal{Z}^B\ . \tag{22}$$

Varying both sides of (21), one finds

$$\delta\lambda_\alpha=\left(L_\alpha{}^\beta+\frac{1}{2}D\delta_\alpha{}^\beta+K_{\alpha\dot\alpha}x^{\dot\alpha\beta}\right)\lambda_\beta-i\rho^{-1}K_{\alpha\dot\alpha}G^{\dot\alpha\dot\beta}\bar\lambda_{\dot\beta} \tag{23}$$

and so

$$\delta x^{\dot\alpha\alpha}\ =\ P^{\dot\alpha\alpha}-x^{\dot\alpha\beta}L_\beta{}^\alpha-\bar L^{\dot\alpha}{}_{\dot\beta}x^{\dot\beta\alpha}+Dx^{\dot\alpha\alpha}+x^{\dot\alpha\beta}K_{\beta\dot\beta}x^{\dot\beta\alpha}-\rho^{-2}K^{\dot\alpha\alpha} \tag{24}$$

$$\delta\rho\ =\ -D\rho-2\rho x\cdot K \tag{25}$$

which are the well-known isometries of anti-de Sitter space.

Geometric invariants may now be constructed by contracting $\mathcal{Z}$ with the $SO(p-2,1)$ metric

$$H_{\dot A}{}^B=\begin{pmatrix}0 & \mathcal{C}^{\dot\alpha\dot\beta}\\ \bar{\mathcal{C}}^{\dot\alpha\dot\beta} & 0\end{pmatrix} \tag{26}$$

where $\mathcal{C}$ is the charge conjugation matrix, so

$$\bar{\mathcal{Z}}_1\cdot\mathcal{Z}_2=\bar\lambda_1\mu_2+\bar\mu_1\lambda_2\ . \tag{27}$$

A natural first guess for a particle action is

$$\mathcal{L}=i\bar{\mathcal{Z}}\partial\mathcal{Z}\ . \tag{28}$$

This matches the kinetic term in (12) if we identify components of λ with the momenta as follows:

$$\begin{aligned}P_{\alpha\dot\alpha}\ &=\ 2\lambda_\alpha\bar\lambda_{\dot\alpha}\\ P_\rho\ &=\ \frac{i}{2\rho^2}\left[\bar\lambda G\bar\lambda-\lambda\bar G\lambda\right]\ .\end{aligned} \tag{29}$$

In (28), however, all the components of λ are independent variables and so must be fixed to physically meaningful values. The first condition is the mass-shell constraint,

$$\frac{1}{2\rho^2}P^2 - \rho^2 P_\rho^2 = \frac{1}{4}\left(\bar{\mathcal{Z}}\mathcal{Z}\right)^2 = M^2 R^2 \ . \tag{30}$$

There exist further independent components of λ for most values of p. These may be fixed by fixing the values of a set of twistor bilinears

$$\phi_i \equiv \bar{\mathcal{Z}} T_i \mathcal{Z} \tag{31}$$

where $(T_i)_{\dot{A}}{}^{B}$ are some constant matrices which transform in the $(\frac{1}{2}, \frac{1}{2})$ of $SO(p-1,2)$. The number of independent ϕ_i that must be set depends on p. In an action, these will be constrained to values m_i. For example, using (30) the mass-shell constraint is $\phi_{T=1} = 2MR$. So the complete twistor action takes the simple form

$$\mathcal{L} = i\bar{\mathcal{Z}}\left(\partial - iu^i T_i\right)\mathcal{Z} - u^i m_i \tag{32}$$

where the u^i are Lagrange multipliers. This action is equivalent to (12). It has several important features:

1. The action is manifestly $SO(p-1,2)$ invariant and has a quadratic kinetic term. It has the structure of a world-line gauge theory with sources. The "gauge fields" u^i are nondynamical since there is no field strength in one dimension.

   This statement can be made somewhat more precise by noting that (28) implies that the Poisson brackets (which will become commutators in the quantized theory) are

   $$\left\{\mathcal{Z}_A, \bar{\mathcal{Z}}^{\dot{B}}\right\}_{PB} = -2iH_A{}^{\dot{B}} \tag{33}$$

   with all other brackets vanishing, and so

   $$\left\{\phi_i, \phi_j\right\}_{PB} = -2i\bar{\mathcal{Z}}\left[T_i, T_j\right]\mathcal{Z} \ . \tag{34}$$

   Since the set of constraints under Poisson brackets forms a Lie algebra, the set of T_i form one as well, and this algebra is invariant under $SO(p-1,2)$. This guarantees that the action (32) indeed has a gauge symmetry.

2. The gauge group contains a $U(1)$ factor corresponding to the mass-shell constraint $T = 1$, $m = 2MR$. The rest of the group may be calculated explicitly for small p by constructing the ϕ_i; they are

p	1	2	3	4	5	6	7
dim $\mathcal{Z}_0$	2	2	4	4	8	8	16
N_ϕ	1	0	1	0	3	2	9
Group	$U(1)^2$	$U(1)$	$U(1)^2$	$U(1)$	$U(1) \times SU(2)$	$U(1)^3$?	$U(1) \times SU(2)^3$?

The final two are conjectured but have not been explicitly calculated.

This is related to the result of [6, 7] for AdS_5. In that case, the 8-component spinors were decomposed into a pair of 4-component spinors of the stability group $H = SO(4,1)$ indexed by $I, J = 1, 2$, and

$$(T_i)_{aI}{}^{bJ} = (\sigma_i)_I{}^J \mathcal{C}_a{}^b \tag{35}$$

(The a, b are $SO(4,1)$ spinor indices)

3. This twistor Lagrangian can be quantized following a procedure similar to that used in [9], leading to solutions which transform in representations of $SO(p-1,2)$.

4. For $i \neq 0$, The ϕ_i may be chosen to be independent of the momenta. In these cases it is not clear what meaning one could assign to a nonzero m_i. The analogous quantities in [6, 7] are all zero.

5. The Lagrange multipliers u^i can be integrated out to give

$$\mathcal{L}'(k) = i\,\bar{\mathcal{Z}}(\partial + iT\cdot m)\mathcal{Z}\big|_k + \int \frac{dq}{2\pi}\,\left(\bar{\mathcal{Z}}T^i\mathcal{Z}\right)\big|_{k+q}\left(\bar{\mathcal{Z}}T_i\mathcal{Z}\right)\big|_{k-q} \tag{36}$$

which is therefore equivalent to (32). (This can also be seen by explicitly resumming Feynman diagrams involving the u^i)

The actions (32) and (36) represent a considerable simplification over their classical counterpart (12). Because they have leading quadratic terms and manifest G-symmetry, their quantum solutions automatically fill out representations of the isometry group. A similar construction can be carried out for an arbitrary coset manifold, or even a supercoset, and (similarly to [9]) can be used to construct string actions on these spaces. Since the known superstring actions are manifestly invariant under the isometries, it is likely that these systems will be amenable to a twistor interpretation which would allow their quantization and analysis, including interactions with Ramond-Ramond and Neveu-Schwarz background fields.

Acknowledgements

The author wishes to thank Piet Claus, Renata Kallosh, Michael Peskin, Joachim Rahmfeld, and Steve Shenker for numerous useful discussions and comments, and the organizers of the TASI-1999 school where part of this work was accomplished. This work was supported in part by an NSF Graduate Research Fellowship and by NSF grant PHY-9870115.

References

[1] J. Maldacena, "The Large N limit of superconformal field theories and supergravity," Adv. Theor. Math. Phys. **2** (1998) 231, hep-th/9711200

[2] S. Gubser, I. R. Klebanov and A. M. Polyakov, "Gauge theory correlators from noncritical string theory," Phys. Lett. **B428** (1998) 105, hep-th/9802109

[3] E. Witten, "Anti-de Sitter space and holography," Adv. Theor. Math. Phys. **2** (1998) 253, hep-th/9802150.

[4] R. Penrose and W. Rindler, *Spinors and Space-time* vol. 2, section 6.3; Cambridge U.P., Cambridge, 1986

[5] A. Ferber, "Supertwistors and Conformal Supersymmetry," Nucl. Phys. **B132** (1978) 55.

[6] P. Claus, J. Rahmfeld and Y. Zunger, "A Simple particle action from a twistor parametrization of AdS(5)," hep-th/9906118.

[7] P. Claus, R. Kallosh and J. Rahmfeld, "BRST quantization of a particle in AdS(5)," hep-th/9906195.

[8] P. K. Townsend, "Supertwistor Formulation of the Spinning Particle," Phys. Lett. **B261** (1991) 65.

[9] P. Claus, M. Gunaydin, R. Kallosh, J. Rahmfeld and Y. Zunger, "Supertwistors as quarks of SU(2, 2|4)," JHEP **05**, 019 (1999) hep-th/9905112.